J. Fluid Mech. (1966), *vol.* 25, *part* 1, *pp.* 165–178
Printed in Great Britain

Magneto-gasdynamic flow over a wedge

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(Received 3 September 1965)

The solution for the flow of a fully ionized gas over a wedge of finite angle is known for the case when the applied magnetic field is aligned with the incident stream. In this flow there are current sheets on the surfaces of the wedge. When the magnetic field is allowed to deviate slightly from the stream, the current sheets may move into the gas and become shock waves. The magnetic fields adjacent to the wedge above and below it have to be matched. A perturbation method is introduced by means of which expressions for the unknown quantities in the different regions may be determined when there are four shocks attached to the wedge. The results give insight into the manner in which the shock-wave pattern develops as the obliquity of the magnetic field to the stream increases. The question of the stability of the shock waves is also examined.

1. Introduction

The basic problems in applications of fluid mechanics are the behaviour of a fluid in motion either with free boundaries or over a solid body. For flow over a solid body a fundamental problem, useful for a general understanding of the properties of the fluid under consideration, is that of a uniform infinite stream impinging upon a semi-infinite wedge. The study of this flow led to important results in conventional gasdynamics, and the same may be expected from the corresponding study in magneto-gasdynamics. Some work has already appeared on the subject. First, Cabannes (1959) presented the solution to the problem of the steady flow of a perfectly conducting fluid over a symmetrical wedge at zero angle of attack when there is an applied magnetic field aligned with the oncoming stream. It is well-known that a magnetic field aligned with the upstream flow at infinity remains aligned everywhere in an inviscid, perfectly conducting fluid.† This problem is the simplest possible extension of gasdynamics. The attached plane stationary magneto-gasdynamic shocks are two in number and symmetrically placed as in gasdynamics. The flow field and the magnetic field inside the wedge remain uncoupled for fields aligned in the flow, so that it is not necessary to specify the conductivity of the wedge. In the absence of a component of magnetic force normal to the surface of the wedge there is no tangential Lorentz force acting on the inviscid fluid particles in contact with the wedge; hence the presence of a current sheet on the wedge surface is permissible and such a surface, in fact, separates the body of moving fluid from the

† This is easy to prove from the basic equations of continuous flow, and an examination of the jump relations across a shock wave (see §3 below) shows quite simply that it remains true across a shock wave.

solid boundary. Expressions were derived by Cabannes for the velocity, density and pressure jumps in terms of the shock angle β and the semi-wedge angle θ. The trigonometric equation for the shock angle was found to be of fifth order in $\tan\beta$ and required to be solved numerically.

The corresponding problem for an applied magnetic field oblique to the stream has received considerable attention from Kogan (1960) who restricted attention to thin wedges and thin aerofoils, for which linearization of the equations is possible and the exercise becomes one involving only the theory of characteristics. Chu & Lynn (1963) considered the problem of the two-dimensional steady flow of an infinitely conducting fluid past a non-conducting wedge with a magnetic field non-aligned with the oncoming stream. By means of a counting procedure they indicated that to obtain sufficient equations to solve for the number of unknown parameters it was required to match the solution for the flow with that found in the wedge. They considered the jump conditions which hold across weak shocks (characteristics) and restricted their analysis to thin wedges. Their prime object in this linear theory was to demonstrate the effect of the coupling of the flows above and below the wedge via the boundary conditions on the magnetic field. In a more recent paper, Mimura (1963) presented a solution to the non-linear problem of the shock-wave configuration on a non-conducting wedge of finite angle in the presence of an incident perfectly conducting steady stream. In this case, however, the magnetic field was applied perpendicular to the uniform flow and was assumed to be weak. He indicated that the flow had to pass through four shock waves, two for the upper surface and two for the lower.

In the following sections, the equations for the problem of flow of fully ionized, inviscid gas past an infinite non-conducting wedge with four attached shock waves are developed in full generality. They are then used to show how flow with four attached shock waves may develop from the solutions of Cabannes when the magnetic field ahead of the wedge becomes oblique to the stream. A method of perturbation is found for small obliquity χ_1 which illustrates how the current sheets, lying along the surfaces of the wedge, move out into the stream to give the additional shock waves. In the wedge a magnetic field, inclined at a finite angle to the wedge axis, is set up. If k_1 is the ratio of Alfvén speed to fluid speed upstream, then for $k_1 < 1$ expressions can be obtained for the quantities in the regions between the second shocks and the wedge surfaces. These have been calculated within the limits imposed by restricting expansions to the second power in χ_1. Perturbation solutions of this kind could not be found for $k_1 \geqslant 1$; however, it has been argued in another paper (Pack & Swan 1965) that for these values of k_1 the shocks found by Cabannes are either unstable or physically unrealizable, and the results obtained here lend support to these views.

2. Statement of the problem

Consider the two-dimensional steady flow of a fully ionized gas, here idealized as a perfect, inviscid fluid of infinite electrical conductivity in irrotational motion over a stationary, semi-infinite, straight-walled, non-conducting symmetric wedge at zero angle of attack to the oncoming stream. Without loss of generality the permeability of the body may be assumed to be the same as that of the incident stream. Diamagnetic effects are ignored and Maxwell's equations are used in their usual form in conjunction with the basic approximations and equations of magneto-gasdynamics. Displacement currents are neglected and the gas is assumed to be electrically neutral. The applied magnetic field of magnitude H_1 is oriented at an angle χ_1 to the incident flow, which has a uniform speed V_1 at infinity upstream and is directed along the axis of the wedge (figure 1).

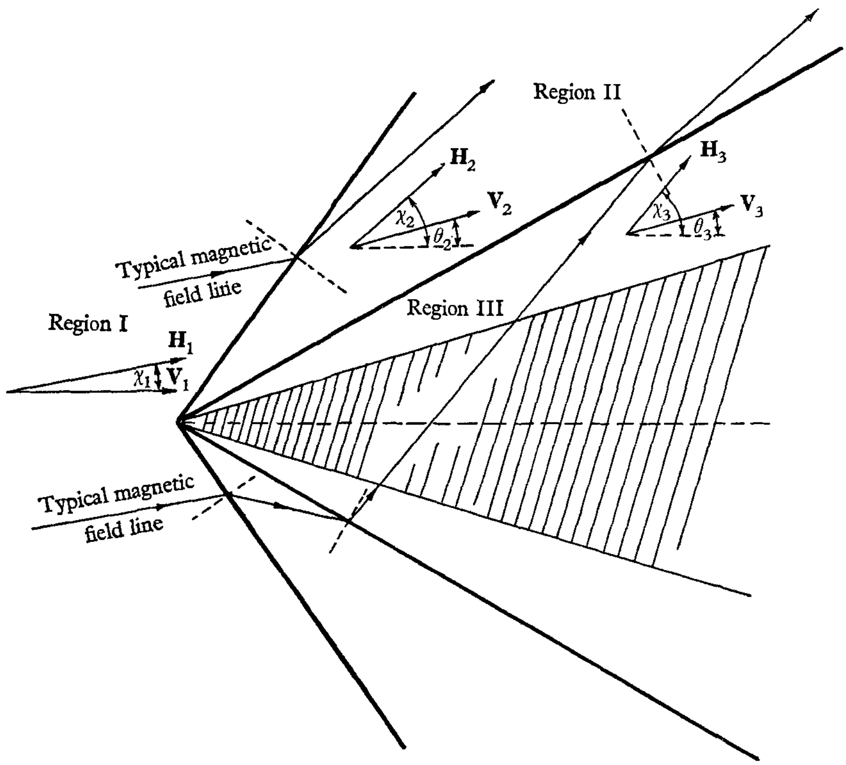

FIGURE 1. Flow over an insulating wedge in an oblique magnetic field: stable four-shock configuration. Two typical magnetic field lines are shown, one of which penetrates the wedge and one does not.

(The suffix 1 refers throughout to conditions upstream of the leading shock wave.) The non-conducting wedge is assumed to be symmetrical with semi-vertex angle θ_3. The restriction to a symmetrical wedge is not necessary (the field is, in any case, unsymmetrical) but leads to some simplification of very complicated equations and makes it easier to draw comparisons with the results of conventional gasdynamics. The two-dimensional flow is assumed to be of restricted type, i.e. the magnetic field is assumed to lie entirely in the plane of the flow, the (x, y)-plane, which is supposed to be normal to the leading edge of the wedge. In this type of two-dimensional flow div $\mathbf{D}$ vanishes, $\mathbf{D}$ being the electric displacement vector, and consequently there can be no distribution of space charge. The addition of a third component of magnetic field independent of z, while making the equations more complicated, is straightforward from a theoretical point of view and will not be considered here. As pointed out by Chu & Lynn (1963), its omission removes from the flow field a pair of Alfvén waves, one above and one below the body.

On the basis of linearized theory, Kogan (1960) has shown that, when the equations of motion are 'fully hyperbolic', there are four real characteristics (other than streamlines) through every point. In a full non-linear theory the characteristics through the apex of the wedge, representing weak discontinuities for the thin wedge, may be expected to be replaced by shock waves or expansion waves, as appropriate. The fluid flow has to be such that the magnetic fields on the surfaces of the wedge are compatible with the field inside the insulating wedge, which is governed by an elliptic differential equation. The lack of alignment in the magnetic field induces different shock and flow patterns on the upper and lower surfaces of the wedge. The solution will be sought, as indicated in figure 1, by the juxtaposition of uniform regions of perfectly conducting fluid separated by shock waves.

3. Basic equations

Rationalized MKS units are used throughout. Under the assumptions made, Maxwell's equations take the form:

$$\operatorname{div}\mathbf{H} = 0, \tag{1}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{j}, \tag{2}$$

and Ohm's law is

$$\mathbf{j} = \sigma(\mathbf{E}+\mu\mathbf{V}\times\mathbf{H}), \tag{3}$$

where $\mathbf{V}$, $\mathbf{H}$ are respectively the velocity and magnetic fields, $\mathbf{j}$ is the current density vector, σ the electrical conductivity, μ the magnetic permeability and $\mathbf{E}$ the electric field.

The equations holding across a plane, stationary shock wave in an infinitely conducting gas are (see, for example, Bazer & Ericson 1959)

$$[H_n] = 0, \tag{4}$$

$$[\rho V_n] = 0, \tag{5}$$

$$[\rho V_n\mathbf{V} + (p+\tfrac{1}{2}\mu H^2)\,\mathbf{n} - \mu H_n\mathbf{H}] = 0, \tag{6}$$

$$[\rho V_n(\tfrac{1}{2}V^2+\gamma p/(\gamma-1)\,\rho+\mu H^2/\rho) - \mu H_n(\mathbf{H}.\mathbf{V})] = 0, \tag{7}$$

$$[V_n\mathbf{H} - H_n\mathbf{V}] = 0, \tag{8}$$

$$\rho V_n[S] \geqq 0, \tag{9}$$

where p is the pressure in the gas, ρ the density, S the specific entropy and γ the ratio of specific heats. Here the suffix n indicates a component normal to the shock wave. The square brackets are used to indicate the change, across the shock wave, in the enclosed quantity.

In the following χ, θ, β and δ are used to denote, respectively, the inclinations of the magnetic field, the fluid velocity, the first and second shock waves to the axis of the wedge. The suffix 2 refers to conditions in region II between the first and second shocks, and the suffix 3 to conditions in region III between the second shock and the wedge surface. Quantities without primes refer to the flow in the upper and primed quantities to the flow in the lower half-plane.

A count of unknown quantities shows that in region II there are the seven unknowns H_2, V_2, ρ_2, p_2, χ_2, θ_2 and β, and in the region III the six unknowns H_3, V_3, ρ_3, p_3, χ_3 and δ. Thus there are thirteen unknowns for the solution on the upper surface; there are also thirteen from the two regions below, giving a combined total of twenty-six unknowns. An examination of the jump relations (4) to (8) shows that across any single shock there are but six independent basic scalar equations; there are therefore twenty-four equations altogether and twenty-six unknowns. Accordingly, we are led to the conclusion that for non-aligned fields the solution in the fluid depends on the solution of the boundary-value problem within the non-conducting wedge. It is easy to show (Swan 1961; Chu & Lynn 1963) that the magnetic field inside the semi-infinite non-conducting wedge must be constant. In consequence, the magnitude and direction of the magnetic field in the wedge are the same on the upper and lower surfaces of the wedge. This result supplies two further relations once the connexion between the values of the magnetic fields in the fluid and the wedge at the interface have been established. This matter is investigated in the next section.

4. Conditions at the fluid-wedge interface

At the interface between the two media the normal component of the magnetic induction is required to be continuous. It has been assumed that there is no change in the permeability across the interface, hence continuity of H_n (the suffix n always indicates the normal component across the interface between two adjoining regions) has to be ensured. The tangential component of **H** may or may not be continuous. If it is not, then a current sheet lies on the interface. In the event that $H_n \neq 0$, the current sheet and magnetic field together produce a Lorentz force acting on the layer of particles in contact with and moving along the wedge. It has been customary to rule this possibility out on the grounds that an inviscid fluid cannot support a surface traction. If this is accepted, then, for $H_n \neq 0$, no current sheet is permissible; in consequence, the tangential component of **H**, and hence **H** itself, must be continuous across the interface. For the problem under consideration this implies that the vector **H** has the same value in the fluid on both the upper and lower surfaces of the wedge. This result supplies the two additional conditions required to bring the number of equations up to the number of unknown quantities and thus to make the problem theoretically soluble. Before proceeding it is useful to point out that the correct tangential boundary condition to be satisfied at the interface is not quite so straightforward as has sometimes been supposed. Stewartson (1960) has discussed at some length the nature of the limiting condition at an interface between solid and fluid as the kinematic viscosity ν in the fluid tends to zero and the electrical conductivity tends to infinity. The jump condition to be satisfied by the magnetic and velocity vectors across the interface is

$$[H_t] = (\rho\nu\sigma)^{\frac{1}{2}}[V_t].$$

In general the values of σ and ν are such that the limits of $\nu\sigma$ may be taken to be zero. Thus $[H_t] = 0$ and consequently $[\mathbf{H}] = 0$ as assumed above. However, in some astrophysical applications the limit may well be finite.† Then a discontinuity in **H** at the interface is necessary and the surface force inevitable. The usual concept of an inviscid fluid must therefore be modified in this case in order to allow a correct representation of the boundary conditions to be made. This state of affairs does not materially affect the solution of the problem under discussion, the result being merely a modification of the two additional conditions on $\mathbf{H}_3$, $\mathbf{H}_3'$ (involving also $\mathbf{V}_3$ and $\mathbf{V}_3'$).

When $H_n = 0$, no difficulties of the above kind arise, the fields in the fluid and wedge being completely uncoupled. Cabannes's problem was therefore capable of solution without any reference to the nature of the wedge or the field inside it.

In what follows it will be assumed for the sake of definiteness and simplicity that

$$\lim_{\substack{\sigma\to\infty \\ \nu\to 0}} \nu\sigma = 0,$$

and that the continuity of **H** across the interface between the fluid and wedge has to be assured.

5. Equations holding across shocks

The first shock on the upper surface

From (4), $H_{1n} = H_{2n}$, or

$$H_2/H_1 = \sin(\beta-\chi_1)/\sin(\beta-\chi_2). \tag{10}$$

From (5), $\rho_1 V_{1n} = \rho_2 V_{2n}$, a statement of the continuity of the mass flux across the shock, and this may be written as

$$\rho_2/\rho_1 = (V_1/V_2)\sin\beta/\sin(\beta-\theta_2). \tag{11}$$

By use of the tangential component of (6) and the equation (10), it follows that

$$\frac{V_2}{V_1} = \frac{\cos\beta}{\cos(\beta-\theta_2)} - k_1^2\frac{\sin(\beta-\chi_1)\sin(\chi_1-\chi_2)}{\sin\beta\cos(\beta-\theta_2)\sin(\beta-\chi_2)}. \tag{12}$$

The quantity k is the ratio of the Alfvén to the flow speed. Equations (11) and (12) may be combined to give

$$\frac{\rho_1}{\rho_2} = \frac{\tan(\beta-\theta_2)}{\tan\beta} - k_1^2\frac{\sin(\beta-\chi_1)\sin(\chi_1-\chi_2)\sin(\beta-\theta_2)}{\sin^2\beta\cos(\beta-\theta_2)\sin(\beta-\chi_2)}. \tag{13}$$

The normal component of (6) combined with (12) leads after some algebra to the result

$$p_2/p_1 + \tfrac{1}{2}\gamma\epsilon_1^2(H_2/H_1)^2 - 1$$
$$= \gamma M_1^2\frac{\sin\beta\sin\theta_2}{\cos(\beta-\theta_2)} + \tfrac{1}{2}\gamma\epsilon_1^2\left\{1 + \frac{2\sin(\beta-\chi_1)\sin(\chi_1-\chi_2)\sin(\beta-\theta_2)}{\sin(\beta-\chi_2)\cos(\beta-\theta_2)}\right\}, \tag{14}$$

where $\epsilon = Mk = b/a$, the ratio of the Alfvén speed to the speed of sound a. As usual, M denotes Mach number.

† The authors are indebted to Prof. K. Stewartson for this observation.

Equation (7) gives

$$\tfrac{1}{2}(1-V_2^2/V_1^2)+k_1^2(1-\rho_1H_2^2/\rho_2H_1^2)+(1-a_2^2/a_1^2)/M_1^2(\gamma-1)$$
$$-k_1^2\{\cos\chi_1-(V_2H_2/V_1H_1)\cos(\chi_2-\theta_2)\}\sin(\beta-\chi_1)/\sin\beta = 0. \qquad (15)$$

When the flow direction θ_2 and the inclination χ_2 of the magnetic field are known, this equation gives an expression for the shock angle β. When the fields are aligned, it reduces to a quintic equation in $\tan\beta$, the equation found by Cabannes and solved by him by numerical methods to give the complete solution for the wedge under this condition.

The component of (8) tangential to the shock gives

$$(V_2/V_1)\sin(\chi_2-\theta_2) = (H_1/H_2)\sin\chi_1. \qquad (16)$$

Substitution of (10) and (12) in (16) leads after some little effort to an equation of the fourth degree in $\tan\beta$, with coefficients involving given quantities and $\tan\chi_2$, $\tan\theta_2$. It may be arranged in the form

$$Q\tan\theta_2 = P, \qquad (17)$$

where

$$P = (\tan\chi_2-\tan\chi_1)\{\tan^2\beta\sec^2\chi_1(\tan\beta-\tan\chi_2)$$
$$+k_1^2\tan\chi_2\sec^2\beta(\tan\beta-\tan\chi_1)^2\}$$

and

$$Q = \{(1-\tan\chi_1\tan\chi_2)\tan^2\beta+(\tan^2\beta-1)\tan\chi_1\tan\beta\}\sec^2\chi_1(\tan\beta-\tan\chi_2)$$
$$+k_1^2(\tan\beta-\tan\chi_1)^2(\tan\chi_2-\tan\chi_1)\sec^2\beta.$$

The six equations (10) and (12)–(16) will form the basis of the analysis in subsequent sections. The remaining 18 shock equations may be derived very simply from these by the transformations indicated below.

The second shock on the upper surface

The angle between the magnetic field and the direction of flow upstream of the second shock on the upper surface is $\chi_2-\theta_2$. By referring all angles to the flow direction in this region, the equations appropriate to this shock follow from those obtained above by means of the substitutions:

$$\delta-\theta_2 \text{ for } \beta,\quad \theta_3-\theta_2 \text{ for } \theta_2,\quad \chi_3-\theta_2 \text{ for } \chi_2 \quad\text{and}\quad \chi_2-\theta_2 \text{ for } \chi_1.$$

The shocks on the lower surface

The equations for the shocks on the lower surface may be obtained directly from those established for the upper surface. The simplest form results if we measure directions downwards from the axis of symmetry in figure 1 and add a prime to the variables β, δ, $\chi_2, \ldots$, to mark quantities in the lower half-plane. The equality of the magnetic field vector for both half-planes ahead of the leading shocks, and also behind the second shocks, is then provided for simply by writing $-\chi_1$ for χ_1 and $-\chi_3$ for χ_3.

As already explained there are 26 dependent variables for which there are 24 equations derived above and two boundary conditions. The transcendental nature of the equations involved renders a direct analytical approach virtually impossible. The equations could be tackled on a fairly large electronic computer, but again the number of parameters suggests that a considerable amount of complicated interpolation would be necessary in order to obtain results. All solutions must of course be subjected finally to the thermodynamic test of non-diminishing entropy laid down by (9), and the choice of branches where two possible shock directions exist has also to be made. In view of this it seemed worthwhile to try to narrow the problem to that of finding how the general configuration begins to develop from a known solution by making a small alteration in some parameter and attempting an analytical approach.

The starting-point chosen was Cabannes's solution for a magnetic field aligned with the stream. The variation introduced was in the direction of the magnetic field upstream of the wedge. The non-alignment of the field provides interesting insight into the adjustment of the fields in the wedge and in the fluid. As the inclination χ_1 of the magnetic field to the stream tends to zero, the configuration has to pass from one in which there is continuity of magnetic field at the interface to one in which a current sheet lies on the surface.

6. Perturbation of the Cabannes solution

Since the field inside the non-conducting wedge is constant, the condition of zero normal component on both upper and lower surfaces of the wedge for aligned fields in the fluid requires that there shall be no magnetic field inside the wedge. Corresponding to the collapse of the second family of characteristics into streamlines in the limiting case of aligned fields, it is to be expected that the second shock wave will fall on to the wedge surface and that this will provide the source of the current sheet appearing in the Cabannes solution. Another way of looking at this is to consider that the magnetic field in region II will orient itself so that $\mathbf{H}_2$ is parallel to the second shock in the limit as $\chi_1 \to 0$, while there will be no magnetic field in region III in this limit. Under these circumstances, on putting $\delta = \chi_2$ in the equation corresponding to (17) for the second shock on the upper surface the condition

$$\tan^2(\chi_2-\theta_2)\{1+\tan(\chi_2-\theta_2)\tan(\theta_3-\theta_2)\}\{\tan(\chi_3-\theta_2)-\tan(\chi_2-\theta_2)\}^2 = 0$$

is obtained, and, if shock angles greater than $\frac{1}{2}\pi$ are ignored, this has the roots

$$\chi_2 = \theta_2 \quad \text{or} \quad \chi_2 = \chi_3.$$

The first of these is consistent with the Cabannes limit; when this is preserved, the analysis outlined below shows that $\chi_2 \neq \chi_3$ in the limit as $\chi_1 \to 0$ and the angle χ_3 is therefore not restricted to approach θ_3 as $\chi_1 \to 0$.

As will be seen below, the values of quantities in regions III and III′ may not be found to order χ_1 unless the perturbations from the Cabannes limits are calculated to order χ_1^2. The following perturbations are therefore introduced, the subscript c representing the (known) Cabannes values:

$$\left.\begin{aligned} H_2/H_1 &= (H_2/H_1)_c + b_2\chi_1 + B_2\chi_1^2, \\ V_2/V_1 &= (V_2/V_1)_c + c_2\chi_1 + C_2\chi_1^2, \\ \rho_1/\rho_2 &= (\rho_1/\rho_2)_c + d_2\chi_1 + D_2\chi_1^2, \\ p_2/p_1 &= (p_2/p_1)_c + e_2\chi_1 + E_2\chi_1^2, \\ \beta &= \beta_c + f_2\chi_1 + F_2\chi_1^2, \\ \chi_2 &= \theta_3 + g_2\chi_1 + G_2\chi_1^2, \\ \theta_2 &= \theta_3 + l_2\chi_1 + L_2\chi_1^2. \end{aligned}\right\} \tag{18}$$

Here $b_2, B_2, c_2, C_2, \ldots, l_2, L_2$ are constants to be determined. A set of six linear equations is obtained by equating terms of first order in χ_1 after substitution of (18) in (10), (11), (12), (14), (15) and (16). The last of these yields at once

$$g_2 - l_2 = (V_1H_1/V_2H_2)_c > 0 \tag{19}$$

(necessarily positive since V and H are scalar resultants). The same procedure in region II′ with (18) modified to read

$$H_2'/H_1' = (H_2/H_1)_c - b_2'\chi_1 + B_2'\chi_1^2, \text{ etc.,}$$

leads to the result

$$g_2' - l_2' = g_2 - l_2 > 0. \tag{20}$$

The flow near the wedge

For the transition to region III the assumptions consistent with the previous analysis are

$$\left.\begin{aligned} \delta &= \chi_2 + f_3\chi_1 + F_3\chi_1^2, \\ \chi_3 &= \alpha + g_3\chi_1 + G_3\chi_1^2, \end{aligned}\right\} \tag{21}$$

where α is the, as yet unknown, orientation of the magnetic field in region III and is $O(1)$. (The particular form chosen for δ, with the perturbation measured from χ_2 instead of θ_3, aids in simplifying the algebra.)

When these relations are substituted into the shock equations and terms of order χ_1 compared, it is found that

$$H_3/H_2 = f_3\chi_1/\sin(\theta_3 - \alpha), \tag{22}$$

$$V_3/V_2 = 1 - k_{2c}^2 f_3/(f_3 + g_2 - l_2), \tag{23}$$

$$\frac{\rho_2}{\rho_3} = \frac{f_3 + g_2}{f_3 + g_2 - l_2}\left\{1 - \frac{k_{2c} f_3}{f_3 + g_2 - l_2}\right\}, \tag{24}$$

$$p_3/p_2 = 1 + \tfrac{1}{2}\gamma\epsilon_{2c}^2, \tag{25}$$

and

$$f_3 + g_2 - l_2 = \pm k_{2c} f_3, \tag{26}$$

where $k_{2c} = (\epsilon_2/M_2)_c$.

In region III′ (in the lower half-plane), the equations corresponding to (21) are

$$\left.\begin{aligned} \delta' &= \chi_2' - f_3'\chi_1 + F_3'\chi_1^2, \\ \chi_3' &= -(\alpha + g_3\chi_1 + G_3\chi_1^2), \end{aligned}\right\} \tag{27}$$

where we have used the result that **H** is constant throughout the non-conducting wedge.

An analysis may be carried out exactly as for region III to obtain formulae for H'_3/H'_2, V'_3/V'_2, etc., corresponding to (22) to (26).

Different cases now arise according as $k_{2c} <$, $=$ or > 1.

6.1. $k_{2c} < 1$

The two values of f_3 given by (26) are negative. A necessary and sufficient condition for the flow in region II to intersect the second shock wave, as it must do in a physically real situation, is that $\delta > \theta_2$ or $f_3 + g_2 - l_2 > 0$. Accordingly, since $f_3 < 0$, the upper sign in (26) has to be dismissed and thus

$$f_3 = -(g_2 - l_2)/(1 + k_{2c}). \tag{28}$$

On substitution of the ratios (22)–(25) into the energy equation, one finds that

$$(f_3 + g_2)(1 + k_{2c})/(-k_{2c} f_3) = \{1 + \tfrac{1}{2}(\gamma - 1)\epsilon_{2c}^2\}/\{1 + \tfrac{1}{2}\gamma\epsilon_{2c}^2\} < 1. \tag{29}$$

This equation yields the value of g_2 and it follows easily that, to a first approximation (independent of χ_1),

$$\left.\begin{aligned} V_3/V_2 &= 1 + k_{2c}, \\ \rho_3/\rho_2 &= \{1 + \tfrac{1}{2}\gamma\epsilon_{2c}^2\}/\{1 + \tfrac{1}{2}(\gamma - 1)\epsilon_{2c}^2\}, \\ \text{and} \quad p_3/p_2 &= 1 + \tfrac{1}{2}\gamma\epsilon_{2c}^2. \end{aligned}\right\} \tag{30}$$

Thus V_3/V_2, ρ_3/ρ_2 and p_3/p_2 are all greater than unity. The last two of these results are required for a shock wave. The first shows that the flow is actually accelerated through the second shock and it is interesting to recall that Kogan (1960) showed that an acceleration could occur across a magneto-gasdynamic shock when he applied linearized theory to the flow past a thin wedge with $\chi_1 = \frac{1}{2}\pi$ (he found such accelerations through the second shocks both above and below the wedge).

The leading shock wave is 'fast' (i.e. the normal components of flow are super-Alfvénic in front of and behind it). The second shock wave satisfies the second law of thermodynamics, and it is easy to show that $V_{n3} < u_{s3}$ (the suffix n here referring to the normal to the second shock), where u_s is the slow magneto-acoustic speed defined below. A further simple calculation shows that $V_{n2} > u_{s2}$ (n again referring here to the second shock) but, to the order of approximation so far achieved,

$$V_{n2} = b_{n2},$$

where $b_n^2 = \mu H_n^2/\rho$ ($b^2 = \mu H^2/\rho$ and is the square of the Alfvén speed). This shows that the second shock wave comes into being with all the properties of a 'switch-off' shock except that there is no normal component of magnetic field in the limit. The true nature of the second shock requires the investigation of the next approximation (at least), and this has been carried out. The algebra is rather long and will not be reproduced here. There are again just sufficient equations and boundary conditions to permit the constant coefficients to be found. Computations have been carried out for $\epsilon_1^2 = 0{\cdot}1$, $M_1 = 1{\cdot}9$, $2{\cdot}0$, $2{\cdot}25$, 2·5, 2·75, 3·0 and $\chi_1 = 2°$ (2°) 12° for a wedge of 20° semi-angle. For these cases the second shock waves have the property that $V_{n2} < b_{n2}$ for small χ_1. Thus the second shock wave on the upper surface is slow. Both fast and slow shocks satisfy the conditions for stability laid down by Akhiezer, Liubarskii & Polovin (1959) and later discussed in considerable detail by Anderson (1963). These conditions are that *one* of the following pairs of inequalities must hold:

(i) $u_{f1} < V_{n1} < \infty, \quad b_{n2} < V_{n2} < u_{f2}$,

(ii) $u_{s1} < V_{n1} < b_{n1}, \quad 0 < V_{n2} < u_{s2}$.

Here the suffices 1, 2 refer respectively to the upstream and downstream sides of whatever shock is being examined and

$$u_f, u_s = [\tfrac{1}{2}(a^2+b^2) \pm \tfrac{1}{2}\sqrt{\{(a^2+b^2) - 4a^2b_n^2\}}]^{\frac{1}{2}},$$

the upper and lower signs referring respectively to u_f, u_s, the so-called fast and slow magneto-acoustic speeds.

Similar considerations applied to the results for the lower half-plane lead to

$$f_3' = -(g_2 - l_2)/(1 - k_{2c}), \tag{31}$$

and
$$V_3'/V_2' = 1 - k_{2c}, \tag{32}$$

while ρ_3'/ρ_2' and p_3'/p_2' have the same values as the corresponding quantities in region III. Equation (32) shows that the flow is decelerated through the second shock, in contrast with what occurs on the upper side of the wedge.

As before the quantities $b_2', c_2', \ldots, l_2'$ can be determined. The first approximation to α is obtained from the equality $H_3 = H_3'$ by writing

$$(H_3'/H_2')(H_2'/H_1) = (H_3/H_2)(H_2/H_1),$$

which gives
$$\{f_3/\sin(\theta_3 - \alpha)\} + \{f_3'/\sin(\theta_3 + \alpha)\} = 0. \tag{33}$$

When f_3, f_3' are replaced by their respective values from (28) and (31) it follows at once that

$$\tan\alpha = (1/k_{2c})\tan\theta_3. \tag{34}$$

This formula gives the inclination of the magnetic field inside the wedge. It shows that $\alpha > \theta_3$ (and incidentally verifies that $H_3/H_2 > 0$ as required).

The second shock on the lower surface has the same kind of limit for $\chi_1 \to 0$ as the one on the upper surface. It can be shown that $V_{n3}' < u_{s3}'$ to $O(1)$ and the computed values satisfy this inequality and also the inequality $u_{s2}' < V_{n2}'$ in all cases. The values obtained for b_{n2}'/V_{n2}' were not always greater than unity—for example, for $M_1 = 2·0$ with $\chi_1 = 8° - 12°$ and for $M_1 = 2·5$, 3·0 for all $\chi_1 > 0$—but it should be pointed out that to the degree of approximation undertaken the values could not include all contributions of $O(\chi_1^2)$, whereas the ratio was so close to unity that terms of this order could alter the conclusions. This was not so on the upper surface, where the computed values exceeded unity by a quantity greater than $O(\chi_1^2)$.

The resolution of the stability question for this second shock on the lower surface is still undecided if the behaviour of the component of magnetic field parallel to the shock is studied. This component must reverse in direction if a shock fs unstable. The computed values of δ', χ_3' indicate marginal stability, but here the

Upper surface

M_1	χ_1 degrees	p_2/p_1	ρ_2/ρ_1	H_2/H_1	χ_2 radians	β radians	θ_2 radians	M_2	δ radians	p_3/p_2	H_3/H_2	χ_3 radians	M_3	ρ_3/ρ_2
2·0	0	2·70	1·99	1·46	0·349	0·927	0·349	1·26	0·349	1·06	0	1·02	—	1·04
	12	2·63	1·96	1·65	0·536	0·926	0·342	1·27	0·377	1·07	0·213	1·05	1·63	1·03
2·5	0	3·04	2·15	1·74	0·349	0·752	0·349	1·70	0·349	1·07	0	1·10	—	1·05
	12	2·99	2·12	1·93	0·493	0·757	0·345	1·71	0·368	1·09	0·157	1·14	2·07	1·03
3·0	0	3·55	2·37	1·99	0·349	0·663	0·349	2·06	0·349	1·08	0	1·15	—	1·05
	12	3·50	2·33	2·18	0·471	0·669	0·346	2·07	0·363	1·09	0·131	1·18	2·47	1·04

Lower surface

M_1	χ_1 degrees	p_2'/p_1'	ρ_2'/ρ_1'	H_2'/H_1'	χ_2' radians	β' radians	θ_2' radians	M_2'	δ' radians	p_3'/p_2'	H_3'/H_2'	χ_3' radians	M_3'	ρ_3'/ρ_2'
2·0	0	2·70	1·99	1·46	0·349	0·927	0·349	1·26	0·349	1·06	0	−1·02	—	1·04
	12	2·83	2·05	1·27	0·168	0·941	0·362	1·22	0·418	1·04	0·280	−1·05	0·899	0·964
2·5	0	3·04	2·15	1·74	0·349	0·752	0·349	1·70	0·349	1·07	0	−1·10	—	1·05
	12	3·11	2·19	1·55	0·206	0·750	0·355	1·70	0·389	1·05	0·197	−1·14	1·33	0·995
3·0	0	3·55	2·37	1·99	0·349	0·663	0·349	2·06	0·349	1·08	0	−1·15	—	1·05
	12	3·59	2·41	1·80	0·228	0·656	0·352	2·09	0·376	1·06	0·159	−1·18	1·69	1·01

Notes. For intermediate values of χ_1 linear interpolation may be used. The results in the columns p_2/p_1 to δ, p_2'/p_1' to δ' have been found to order χ_1^2 and those in columns p_3/p_2 to ρ_3/ρ_2, p_3'/p_2' to ρ_3'/ρ_2' to order χ_1. The results will be correct overall to 2 significant figures.

TABLE 1. Results for magneto-gasdynamic flow over a wedge with $\theta_3 = 20° = 0·349$ radians, $\epsilon_1^2 = 0·1$

calculation lacks terms of $O(\chi_1^2)$ in the value of χ_3'. For full second-order terms to be obtained some of the expansions used would need to be calculated to $O(\chi_1^3)$; this would have involved an excessive amount of computation.†

In the table are given the results of computations based on the series expansions for $M_1 = 2{\cdot}0$, $2{\cdot}5$, and $3{\cdot}0$ and for $\chi_1 = 0°$ and $12°$. Values for intermediate values for χ_1 may be found by linear interpolation.

The results show that as χ_1 increases the pressure ratio across the second shock on the lower surface falls from its limiting values while on the upper surface it rises above this value. The density ratio across the second shock on the lower surface also falls as χ_1 increases, and indeed drops a little below unity for the two lower Mach numbers. This is contrary to the requirements for a shock wave, but the ratio can only be calculated to $O(\chi_1)$ and the fall is therefore not conclusive.‡

If some of the shock-wave configurations are in fact unstable, some break-up of the stipulated components of the flow into more complicated wave systems is to be expected. It has also to be pointed out that the investigation of this paper is restricted to flows with four shock waves. The possibility of the occurrence of flows with both shocks and expansion waves has not been considered.

6.2. $k_{2c} = 1$

When $k_{2c} = 1, f_3 + g_2 - l_2 = \pm f_3$. The upper sign may be rejected because of the result (19). By inspection the lower sign is found to be admissible. However, since also $f_3' + g_2' - l_2' = \pm f_3'$ the upper sign here may be rejected for the same reason as before (use (20)). The lower sign gives $f_3' < 0$. But for the flow in region II′ to intersect the shock, $f_3' + g_2' - l_2'$ must be negative; there is thus a contradiction. Therefore, with this value of k_{2c} a shock-wave solution of the type sought cannot be found. It has been demonstrated by Pack & Swan (1965) that this value of k_{2c} is associated with shock waves that are physically unstable in the Cabannes problem.

6.3. $k_{2c} > 1$

Mathematical consistency now demands that $f_3 = (g_2 - l_2)/(k_{2c} - 1)$ and

$$f_3' = -(g_2 - l_2)/(k_{2c} + 1).$$

This implies that $f_3' + g_2' - l_2' > 0$, with the consequence that the flow behind the first (lower) shock wave cannot meet the postulated second shock. There cannot therefore be two shock waves on the lower surface of the wedge. Pack

† After this paper was written it was discovered that a group at the Mathematics Research Center, United States Army, University of Wisconsin, U.S.A., working under the direction of Dr L. B. Rall had devised a program capable of dealing with the full system of 24 equations. Mrs Julia Gray kindly solved the equations for the wedge of 20° semi-angle with $\epsilon_1^2 = 0{\cdot}1$, $M_1 = 2{\cdot}5$ and $\chi_1 = 12°$. Her values show the second shock on the lower surface to be stable. They differ from the values computed for this paper by no more than one unit in the second significant figure (and in the majority of cases agree to this level of accuracy), even for quantities computed here only to $O(\chi_1)$, except for χ_2' which has a three unit difference, corresponding to $O(\frac{1}{2}\chi_1^2)$.

The authors are indebted to Dr Rall for permission to quote these results.

‡ In Mrs Gray's results $\rho_3'/\rho_2' > 1$.

& Swan (1965) have shown that this value of k_{2c} is associated with shock waves which, if they exist, are either not physically stable or are unacceptable on other grounds.

One of us (G. W. S.) was indebted to the Sir James Caird Scholarship Trust for a Junior Scholarship in Aeronautics during the early stages of this work. Latterly, the work was sponsored by the Air Force Office of Scientific Research, OAR, through the European Office, Aerospace Research, United States Air Force, under Grants nos. EOAR 64–6 and 65–58.

The computations were carried out on the Ferranti Sirius computer at the University of Strathclyde.

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